A New Mass Formula for NG Bosons in QCD

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July 1995 - hep-ph/9507268

Abstract

An often used mass formula for Nambu-Goldstone (NG) bosons in QCD, such as the pions, involves the condensate $<\overline{q}q>$, f_π and the quark current masses. We argue, within the context of the Global Colour Model to QCD, that this expression is wrong. Analysis of the interplay between the Dyson-Schwinger equation for the constituent quark effect and the Bethe-Salpeter equation for the NG boson results in a new mass formula.

PACS numbers: 12.38.Lg, 11.30.Qc, 11.30.Rd, 14.40.Aq, 11.10.St, 12.38.Aw

Keywords: PCAC, Nambu-Goldstone boson, pion mass formula, Quantum Chromodynamics

The computation of the low energy properties of QCD is a difficult non-perturbative problem in quantum field theory. However one species of hadron, the (almost) Nambu-Goldstone (NG) bosons, such as the pions, have always played a key role. Because they

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are directly associated with the dynamical breaking of chiral symmetry their properties are strong indicators of the nature of the underlying quark-gluon dynamics in QCD. However the small current masses of the u and d quarks means that the pions are not strictly massless, as for true NG bosons, but acquire small masses. They are therefore also significant as low mass hadronic excitations.

One expects that there should be some perturbative expression for the pion mass in terms of the quark current masses which is built upon the underlying non-perturbative chiral-limit quark-gluon dynamics. While the relation of the low pion mass to the breaking of chiral symmetry dates back to the current algebra era and PCAC [1], the often used implementation in QCD has the form,

$$M_\pi^2 = \frac{(m_u + m_d)\rho}{f_\pi^2} \tag{1}$$

where the integral $\rho =< \overline{q}q >$ is the so called condensate parameter. For $N_c = 3$

$$\rho = N_c tr(G(x=0)) = 12 \int \frac{d^4q}{(2\pi)^4}\sigma_s(q^2), \tag{2}$$

and f_π is the usual pion decay constant. In (2) $\sigma_s(s)$ is the chiral limit ($m \to 0$) scalar part of the constituent quark propagator

$$G(q) = (iA(s;m)q.\gamma + B(s;m) + m)^{-1} = -iq.\gamma\sigma_v(s;m) + \sigma_s(s;m). \tag{3}$$

We note that the expression for ρ in (2) is divergent in QCD, because for large $s \to \infty$ $B(s)$ decreases like $1/s\ln[s/\Lambda^2]^{1-\lambda}$ where $\lambda = 12/(33 - 2N_f)$ and Λ is the QCD scale parameter. Some integration cutoff is usually introduced. The values of m and $< \overline{q}q >$ are then usually quoted as being relative to some cutoff momentum, often $1GeV$. An alternative approach [2] is to use finite energy sum rules and Laplace sum rules.

Here we present a new analysis of the chiral symmetry breaking in the Global Colour Model (GCM) to QCD. We extract a new expression for the pion mass, which essentially replaces (2). Nevertheless the new result, see equation (22) below, is very similar to (1) except that it contains a naturally arising cutoff function $c(s)$, and also a dynamical enhancement function $\epsilon_s(s)$ for the quark current mass m. There is also a factor of 2 difference. In the Appendix we give an example of the type of incorrect

analysis that leads to the form in (2). The new mass formula implies that the pion mass is dominated by IR processes, and not UV processes as in (2).

An overview and an insight into the nature of the non-perturbative low energy hadronic regime of QCD is provided by the functional integral hadronization of QCD [4, 5]. This amounts to a dynamically determined change of functional integration variables, from quarks and gluons, to bare hadrons

$$\int \mathcal{D}\overline{q}\mathcal{D}q\mathcal{D}A exp(-S_{qcd}[A,\overline{q},q]+\overline{\eta}q+\overline{q}\eta) \approx$$
$$\int \mathcal{D}\pi\mathcal{D}\overline{N}\mathcal{D}N...exp(-S_{had}[\pi,...,\overline{N},N,..]+J_{\pi}[\overline{\eta},\eta]\pi+..) \tag{4}$$

The final functional integration over the hadrons gives the hadronic observables, and amounts to dressing each hadron by, mainly, lighter mesons. This functional integral transformation cannot yet be done exactly. The basic insight is that the quark-gluon dynamics, on the LHS of (4), is fluctuation dominated, whereas the RHS is not, and for example the meson dressing of bare hadrons is known to be almost perturbative. In performing the change of variables essentially normal mode techniques are used [4]. In practice this requires detailed numerical computation of the gluon propagator, quark propagators, and meson and baryon propagators. The mass-shell states of the latter are determined by covariant Bethe-Salpeter and Faddeev equations. The Faddeev computations are made feasible by using the diquark correlation propagators, which must also be determined.

The first and easiest formal transformation results from doing the gluon integrations, leaving an action for quarks of the form

$$S[\overline{q},q]=\int \overline{q}(x)(-\gamma.\partial+\mathcal{M})q(x)+\frac{1}{2}\int j^a_\mu(x)j^a_\nu(y)D_{\mu\nu}(x-y)+\frac{1}{3!}\int j^a_\mu j^b_\nu j^c_\rho D^{abc}_{\mu\nu\rho}+... \tag{5}$$

where $j^a_\mu(x)=\overline{q}(x)\frac{\lambda^a}{2}\gamma_\mu q(x)$. The GCM is a model field theory for QCD based on a truncation of $S[\overline{q},q]$ in which the higher order n-point ($n\geq 3$) functions are neglected, and only the gluon 2-point function $D_{\mu\nu}(x-y)$ is retained.

The GCM is thus a quantum field theory that can also be considered to be defined by the action

$$S_{gcm}[\overline{q},q,A^a_\mu]=\int\Big(\overline{q}(x)(-\gamma.\partial+\mathcal{M}+iA^a_\mu\frac{\lambda^a}{2}\gamma_\mu)\delta(x-y)q(y)+$$
$$+\frac{1}{2}A^a_\mu(x)D^{-1}_{\mu\nu}(i\partial)\delta(x-y)A^a_\nu(y)\Big) \tag{6}$$

where the matrix $D^{-1}_{\mu\nu}(p)$ is the inverse of $D_{\mu\nu}(p)$, which in turn is the Fourier transform of $D_{\mu\nu}(x)$. This action has a global colour symmetry. The GCM is thus analogous to QED except for colour currents and the non-quadratic phenomenological form for $D^{-1}_{\mu\nu}(p)$ in the pure gluon sector. The determination of the best-fit phenomenological form for $D_{\mu\nu}(p)$ was reported in [7] using the separable expansion technique to facilitate numerical computations and the fitting to some meson data.

Having made the GCM truncation in (5) it is possible to proceed further and to transform [4] the quark functional integrations into the hadronic functional integrations, as in (4). If the additional approximation $D_{\mu\nu}(x-y) \to g\delta_{\mu\nu}\delta(x-y)$ is made in (6), i.e. a contact coupling of the quark currents, then the NJL type models are obtained. If in (4) a derivative expansion of the complete non-local hadronic effective action is performed, then the Chiral Perturbation Theory (CPT) phenomenology is obtained. However in the GCM, with appropriate $D_{\mu\nu}(x)$, all computations are finite and no cutoffs or renormalisation procedures are used. As well, using a mean field approximation, the soliton phenomenology for the baryons may be derived [3], and has been studied in [6].

In analogy with QED the GCM involves the determination of various coupled equations for the propagators. In the approximation often employed (which is motivated by the functional integration analysis of (6), see [4]) the first equation is the Dyson-Schwinger (DSE) equation for the constituent quark propagator (the so-called vacuum equation of the GCM [3, 4]),

$$B(p^2;m) = \frac{16}{3}\int \frac{d^4q}{(2\pi)^4} D(p-q).\frac{B(q^2;m)+m}{q^2A(q^2;m)^2+(B(q^2;m)+m)^2}, \tag{7}$$

$$[A(p^2;m)-1]p^2 = \frac{8}{3}\int \frac{d^4q}{(2\pi)^4} q.pD(p-q).\frac{A(q^2;m)}{q^2A(q^2;m)^2+(B(q^2;m)+m)^2}, \tag{8}$$

using a Feynman-like gauge, $D_{\mu\nu}(p) = \delta_{\mu\nu}D(p)$, and the perturbative quark-gluon vertex function. The Landau gauge can also be used; see [7] for comparison.

Using Fourier transforms (7) may be written in the form, here for $m=0$,

$$D(x) = \frac{3}{16}\frac{B(x)}{\sigma_s(x)}, \tag{9}$$

which implies that knowledge of the quark propagator determines the effective GCM gluon propagator. Multiplying (9) by $B(x)/D(x)$, and using Parseval's identity for the

RHS, we obtain the identity

$$\int d^4x \frac{B(x)^2}{D(x)} = \frac{16}{3}\int \frac{d^4q}{(2\pi)^4} B(q)\sigma_s(q). \tag{10}$$

The second basic equation is the Bethe-Salpeter equation (BSE) for the pion mass-shell state at the level of approximation that matches (7) and (8) in the GCM analysis [4, 8]

$$\Gamma^f(p,P) = \frac{8}{3}\int \frac{d^4q}{(2\pi)^4} D(p-q) tr_{SF}(G_+T^gG_-T^f)\Gamma^g(q,P) \tag{11}$$

where $G_\pm = G(\pm q - \frac{P}{2})$. This BSE is for isovector NG bosons, and only the dominant $\Gamma = \Gamma^f T^f i\gamma_5$ amplitude is retained (see [8] for discussion); the spin trace arises from projecting onto this dominant amplitude. Here $\{T^b, b = 1,..,N_F^2-1\}$ are the generators of $SU(N_F)$, with $tr(T^fT^g) = \frac{1}{2}\delta_{fg}$.

The BSE (11) is an implicit equation for the mass shell $P^2 = -M^2$. It has solutions *only* in the time-like region $P^2 \leq 0$. Fundamentally this is ensured by (7) and (8) being the specification of an absolute minima of an effective action after a bosonisation [4]. Nevertheless the loop momentum is kept in the space-like region $q^2 \geq 0$; this mixed metric device ensures that the quark and gluon propagators remain close to the real space-like region where they have been most thoroughly studied. Very little is known about these propagators in the time-like region $q^2 < 0$. The GCM gives a detailed description of the pion properties, including its coupling to other states, in the language of effective non-local actions [4].

The non-perturbative quark-gluon dynamics is expressed here in (7) and (8). Even when $m = 0$ eqn.(7) can have non-perturbative solutions with $B \neq 0$. This is the dynamical breaking of chiral symmetry.

When $m = 0$ eqn.(11) has a solution for $P^2 = 0$; the Goldstone theorem effect. For the zero linear momentum state $\{P_0 = 0, \vec{P} = \vec{0}\}$ it is easily seen that eqn.(11) reduces to eqn.(7) with $\Gamma^f(q,0) = B(q^2)$. When $\vec{P} \neq \vec{0}$ then $\Gamma^f(q,P) \neq B(q)$, and (11) must be solved for $\Gamma^f(q,P)$.

We shall now determine an accurate expression for the mass of the pion when $m \neq 0$. This amounts to finding an analytic solution to the BSE (11), when the constituent quark propagators are determined by (7) and (8). The result will be accurate to order m.

For small $m \neq 0$ we can introduce the Taylor expansions in m

$$B(s;m) + m = B(s) + m.\epsilon_s(s) + O(m^2), \tag{12}$$

$$A(s,m) = A(s) + m.\epsilon_v(s) + O(m^2). \tag{13}$$

For large space-like s we find that $\epsilon_s \to 1$, but for small s we find that $\epsilon_s(s)$ can be significantly larger than 1 (see Fig.1). This is a dynamical enhancement of the quark current mass by gluon dressing in the infrared region. Even in the chiral limit the quark running mass $M(s) = B(s)/A(s)$ is essential for understanding any non-perturbative QCD quark effects. At $s = 0.3GeV^2$ we find [7] that $M(s) \approx 270MeV$.

Because the pion mass M_π is small when m is small, we can perform an expansion of the P_μ dependence in the kernel of (11). Since the analysis is Lorentz covariant we can, without loss of validity, choose to work in the rest frame with $P = (iM_\pi, \vec{0})$, giving, for equal mass quarks for simplicity,

$$\Gamma(p) = \frac{2}{9}M_\pi^2 \int \frac{d^4q}{(2\pi)^4} D(p-q) I(s)\Gamma(q) +$$

$$+ \frac{16}{3}\int \frac{d^4q}{(2\pi)^4} D(p-q) \frac{1}{s(A(s)+\epsilon_v(s).m)^2 + (B(s)+m.\epsilon_s(s))^2}\Gamma(q) + ..., \tag{14}$$

where

$$I(s) = 6\left(\sigma_v^2 - 2(\sigma_s\sigma_s' + s\sigma_v\sigma_v') - s(\sigma_s\sigma_s'' - (\sigma_s')^2) - s^2(\sigma_v\sigma_v'' - (\sigma_v')^2\right). \tag{15}$$

By using Fourier transforms the integral equation (14), now with explicit dependence on M_π, can be expressed in the form of a variational mass functional,

$$M_\pi[\Gamma]^2 = -\frac{24}{f_\pi[\Gamma]^2}\int \frac{d^4q}{(2\pi)^4} \frac{\Gamma(q)^2}{s(A(s)+\epsilon_v(s).m)^2 + (B(s)+m.\epsilon(s))^2} + \frac{9}{2f_\pi[\Gamma]^2}\int d^4x \frac{\Gamma(x)^2}{D(x)} \tag{16}$$

in which

$$f_\pi[\Gamma]^2 = \int \frac{d^4q}{(2\pi)^4} I(s)\Gamma(q)^2. \tag{17}$$

The functional derivative $\delta M_\pi[\Gamma]^2/\delta\Gamma(q) = 0$ reproduces (14). The mass functional (16) and its minimisation is equivalent to the pion BSE in the near chiral limit. To find an estimate for the minimum we need only note that the change in M_π^2 from its chiral limit value of zero will be of 1st order in m, while the change in the zero linear momentum frame $\Gamma(q)$ from its chiral limit value $B(q^2)$ will be of 2nd order in m.

Hence to lowest order in m we have that the pion mass is given by

$$M_\pi^2 = \frac{48m}{f_\pi [B]^2} \int \frac{d^4q}{(2\pi)^4} \frac{\epsilon_s(s)B(s) + s\epsilon_v(s)A(s)}{sA(s)^2 + B(s)^2} \frac{B(s)^2}{sA(s)^2 + B(s)^2}$$

$$- \frac{24}{f_\pi [B]^2} \int \frac{d^4q}{(2\pi)^4} \frac{B(s)^2}{sA(s)^2 + B(s)^2} + \frac{9}{2 f_\pi [B]^2} \int d^4x \frac{B(x)^2}{D(x)} + O(m^2) \tag{18}$$

However the pion mass has been shown to be zero in the chiral limit. This is confirmed as the two $O(m^0)$ terms in (18) cancel because of the identity (10). Note that it might appear that f_π would contribute an extra m dependence from its kernel in (15). However because the numerator in (16) is already of order m, this extra contribution must be of higher order in m.

Hence we finally arrive at the analytic expression, to $O(m)$, for the NG boson $(\text{mass})^2$ from the solution of the BSE in (11) which includes the non-perturbative gluon dressing to give constituent quarks

$$M_\pi^2 = \frac{2m\rho_{eff}}{f_\pi^2} \tag{19}$$

where

$$\rho_{eff} = 24 \int \frac{d^4q}{(2\pi)^4} \left(\epsilon_s(s)\sigma_s(s) + s\epsilon_v(s)\sigma_v(s)\right) c(s) \tag{20}$$

defines an effective condensate parameter ρ_{eff}. The ρ_{eff} integrand involves a naturally arising function

$$c(s) = \frac{B(s)^2}{sA(s)^2 + B(s)^2}, \tag{21}$$

which acts as a smooth cutoff function. It is this function which causes the pion mass to be IR dominated. Of course the determination of $B(s)$ and $A(s)$ by (7) and (8) involves both IR and UV momenta. As well ρ_{eff} contains a contribution from the vector part of the chiral-limit quark propagator. Because this vector part makes a small contribution we can approximate (20) by the form

$$\rho_{eff} = 24 \int \frac{d^4q}{(2\pi)^4} \epsilon_s(s) c(s) \sigma_s(s) \tag{22}$$

This result contradicts (2). Significantly the factor $c(s)$ makes (22) convergent when $B(s)$ has the QCD determined asymptotic form discussed after (3). The ϵ enhancement function is important to the numerical values but does not affect the convergence analysis because asymptotically $lim_{s\to\infty}\epsilon_s(s) = 1$.

We now give examples of the functions occuring in (22). In ref. [7] the GCM phenomenological chiral-limit quark propagator and the corresponding gluon propagator were obtained by fitting a number of meson observables to meson data. From these forms the cutoff function $c(s)$ and the enhancement function $\epsilon_s(s)$ can be computed. The resulting functions are shown in Fig.1. We note that the value of $\epsilon_s(s) \approx 4$ implies that in the infrared region, appropriate to the internal dynamics of hadrons, the quark current mass of $\sim 6MeV$ is enhanced by gluon dressing to some $24MeV$. Of course the major effect is the chiral-limit constituent mass of some $270MeV$. We compute using (22) that $\rho_{eff} = (0.233GeV)^3$, and from (17) that $f_\pi = 93.0MeV$. With $m = 6.5MeV$ (22) gives $M_\pi = 138.5MeV$ without cutoffs or renormalisation procedures. We note that these numerical results are actually close to the usual values using the forms in (1) and (2) with the traditional cutoff of $1GeV$. This may be incidental to the particular parameterisation used for the phenomenologcal GCM propagators used in [7].

It would be interesting to see if the corrected form in (22) can be rigorously obtained within QCD.

Appendix: Here we present one form of the flawed analysis that leads to the incorrect result in (2). Consider the mass functional in (16). Suppose we make the ansatz that

$$\Gamma_\pi(q) = B(q^2; m). \tag{23}$$

From (7) we obtain by Fourier transforms and Parseval's identity, in analogy to (10),

$$\int d^4x \frac{B(x;m)^2}{D(x)} = \frac{16}{3} \int \frac{d^4q}{(2\pi)^4} B(q;m) \frac{B(q;m) + m}{q^2 A(q;m)^2 + (B(q;m) + m)^2}. \tag{24}$$

Using (23) and (24) the mass functional (16) then gives

$$\begin{aligned} M_\pi^2 = -\frac{24}{f_\pi^2} \int \frac{d^4q}{(2\pi)^4} \frac{B(q;m)^2}{s(A(s) + \epsilon_v(s).m)^2 + (B(s) + m.\epsilon(s))^2} + \\ \frac{24}{f_\pi^2} \int \frac{d^4q}{(2\pi)^4} \frac{B(q;m)(B(q;m) + m)}{q^2 A(q;m)^2 + (B(q;m) + m)^2}. \end{aligned} \tag{25}$$

$$= \frac{24}{f_\pi^2} \int \frac{d^4q}{(2\pi)^4} \frac{B(q;m)m}{q^2 A(q;m)^2 + (B(q;m) + m)^2}. \tag{26}$$

$$= \frac{24}{f_\pi^2} \int \frac{d^4q}{(2\pi)^4} \frac{B(q)m}{q^2 A(q)^2 + (B(q))^2}. \tag{27}$$

to $O(m)$, and this gives eqns(1) and (2). The flaw in this argument is to overlook that in perturbation theory eigenvalues change at 1st order and eigenvectors change at 2nd

order of the perturbation parameter. The ansatz (23) actually gives $\Gamma_\pi(q)$ a hidden 1st order dependence on m, and is thus incorrect. Essentially the same error has been made in the functional integral formulation of the GCM, where the ansatz (23) arises through the choice of the non-local meson fluctuation. In all GCM papers ρ should be replaced by ρ_{eff}.

RTC acknowledges useful conversations with C. J. Burden, C. D. Roberts, P. C. Tandy and M. Thomson. C. D. Roberts [9] has independently arrived at similar results to those reported here.

Figure Caption

Fig.1 Shows (a) the scalar enhancement function $\epsilon_s(s)$, (b) the integrand $s\epsilon_s(s)c(s)\sigma_s(s)$ (from eqn.(22)) of the effective condensate parameter ρ_{eff} (in arbitrary units), and (c) the cutoff function $c(s)$.

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